Classical lambda calculus in modern dress

J. M. E. Hyland

November 27, 2012

Abstract

Recent developments in the categorical foundations of universal algebra have given fresh impetus to an understanding of the lambda calculus coming from categorical logic: an interpretation is a semi-closed algebraic theory. Scott's representation theorem is then completely natural and leads to precise theorems showing the essential equivalence with more familiar notions. Simple abstract proofs of fundamental results in the semantics of the lambda calculus are given.

1 Introduction

The λ-calculus is one of the great discoveries of logic in the 20th century, but the question of its semantics has proved vexed. Barendregt's monumental text [1] offered a variety of approaches and it is telling that under the influence of Scott [20] the treatment of the semantics was largely rewritten for the revised edition [2]. I offer a fresh view of the semantics. I hope that it shows that the λ-calculus is a simple natural object of mathematical study.

The definition explained here is that an interpretation[1] of the lambda calculus is an algebraic theory equipped with semi-closed structure: I call such a structure a λ-theory. One benefit of this view, characteristic of categorical logic, is that the inductive definition of an interpretation is done once and for all in the abstract setting: there is no further need for it in individual cases. An algebraic theory is admittedly a slightly more complicated mathematical structure than one expects in semantics, but interpreting abstraction

[1] I use the neutral term interpretation in order to respect as far as possible established usage of the terms λ-algebra and λ-model.

involves handling free variables and an algebraic theory makes them explicit. Moreover the definition is effective when it comes to examples. Common interpretations are naturally presented as semi-closed algebraic theories so there is a real gain. A further benefit of making the notion of theory central is that it leads quickly to some fundamental aspects of the theory. In particular I show that Scott's interpretation by reflexive objects in cartesian closed categories arises quite naturally.

The initial λ theory, denoted Λ, has a presentation by the syntax of the λ-calculus. As an algebraic theory it has algebras, and a Λ-algebra is essentially a clean version of valuation interpretations of the λ calculus. The initial Λ-algebra is the closed term interpretation traditionally written Λ_0. The extraordinary fact is that every λ-theory is the theory of extensions of some Λ-algebra. This gives a very tight equivalence between the categories of λ-theories and that of Λ-algebras. I want to call this the Fundamental Theorem of the Lambda Calculus.

I started writing this paper with a section on Combinatory Algebra and λ-calculus, justifying the λ-theory definition. Hints as to what is involved are in [12] and especially [21], but I have had to cut that. and other things. I am particularlly sorry not to make more links with the universal algebra aspects of the λ-calculus pioneered by Antonino Salibra and his co-workers, see [19].

The notion of interpretation of the lambda calculus which I propose derives from an approach to semantics pioneered long ago[2] by Andy Pitts in which one systematically takes contexts seriously. That point of view seems compelling today in view of the good foundations for notions of algebra provided by Kleisli Bicategories [10] but I restrain myself from discussing that. In particular I say nothing about the theory of variable binding initiated in [11] and extended variously for example in [22]. This is an important direction but is not needed for a first appreciation of the λ-calculus.

All the same what I present is abstract enough. I hope that readers will find what I present suggestive for further research. I have written this account to honour Corrado Böhm, the creator of the technique [6] which most influenced my understanding of the subject. In this foundational treatment I cannot get as far as Böhm's Theorem but I close the paper with suggestions for the future including reflections on Böhm's legacy.

[2]His generally unrecognised influence is already apparent in [5] for example.

2 Algebraic theories

2.1 Algebraic theories as cartesian operads

An algebraic theory is a theory of equality on terms. A clean mathematical expression of the idea is via multicategories or operads, so abstractly via monads in some Kleisli bicategory [10]. I only need the concrete version of a one object cartesian multicategory or cartesian operad. Write **Sets** for the category of sets and **F** for a standard skeleton of finite sets.

Definition 2.1 *An* algebraic theory $\mathcal{T}$ *is first a functor* $\mathcal{T} : \mathbf{F} \to \mathbf{Sets}$*: so we have sets* $\mathcal{T}(n)$ *of* n*-ary multimaps with variable renamings. In addition,* $\mathcal{T}$ *is equipped with projections* $\mathrm{pr}_1, \cdots \mathrm{pr}_n \in \mathcal{T}(n)$ *including as special case the identity* $\mathrm{id} \in \mathcal{T}(1)$*. Finally there are compositions* $\mathcal{T}(m) \times \mathcal{T}(n)^m \to \mathcal{T}(n)$ *which are associative, unital, compatible with projections and natural in* n *and* m *(some would say dinatural in* m*).*

Writing $\Gamma \vdash t$ for t a term with variable declaration Γ, the definition encapsulates the basic principles of term formation

$$\frac{}{\Gamma \vdash x} \qquad\qquad \frac{\Gamma \vdash t \quad \Delta \vdash s_1, \cdots \Delta \vdash s_n}{\Delta \vdash t(\mathbf{s})}$$

with x declared in Γ and $t(\mathbf{s})$ the result of substituting the string of terms $\mathbf{s}$ in t. Experts will be aware of other possible formulations of the notion of algebraic theory. The basic syntactic rule of equality, the substitution of equals for equals, is implicit in all presentations.

Definition 2.2 *A map* $F : \mathcal{S} \to \mathcal{T}$ *of algbraic theories consists of a natural transformation with components* $F_n : \mathcal{S}(n) \to \mathcal{T}(n)$ *preserving projections and composition. This gives a category of algebraic theories.*

2.2 Algebras for theories

Very general notions of interpretation for an algebraic theory are supported by [10] but we only need the basic interpretation in the category of sets.

Definition 2.3 *An algebra for an algebraic theory* $\mathcal{T}$ *is a set* A *with an associative unital action* $\mathcal{T}(n) \times A^n \to A$ *of* $\mathcal{T}$*, natural in* n*.*
If A *and* B *are* $\mathcal{T}$*-algebras then a homomorphism from* A *to* B *consists of a map* $f : A \to B$ *respecting the actions in the sense that the evident diagram commutes. This gives a category of* $\mathcal{T}$*-algebras denoted* $\mathrm{Alg}(\mathcal{T})$*.*

The compositions $\mathcal{T}(m) \times \mathcal{T}(n)^m \to \mathcal{T}(n)$ give each $\mathcal{T}(n)$ the structure of a $\mathcal{T}$-algebra, and a Yoneda Lemma argument shows the following

Proposition 2.4 *The algebra $\mathcal{T}(n)$ is the free $\mathcal{T}$-algebra on n generators; for $A \in \mathrm{Alg}(\mathcal{T})$ we have $\mathrm{Alg}(\mathcal{T})(\mathcal{T}(n), A) \cong A^n$ natural in A and n.*

In particular note that $\mathcal{T}(0)$ is the initial $\mathcal{T}$-algebra. We shall not need more general free algebras we shall want to understand the free extension $A[n]$ of a $\mathcal{T}$-algebra A by n indeterminates. That is the coproduct $\mathcal{T}(n) + A$ or more usefully for us the pushout $\mathcal{T}(n) +_{\mathcal{T}(0)} A$ over the initial algebra.

Let $F : \mathcal{S} \to \mathcal{T}$ be a map of algebraic theories. Evidently F takes actions of $\mathcal{T}$ to actions of $\mathcal{S}$ and we have a functor $F^* : \mathrm{Alg}(\mathcal{T}) \to \mathrm{Alg}(\mathcal{S})$. Take B a $\mathcal{T}$-algebra with free extension $B[n]$. The induced $\mathcal{S}$-algebra $F^*(B[n])$ is equipped with maps from $F^*(\mathcal{T}(n))$ and F^*B and so (using the canonical map $\mathcal{S}(n) \to F^*\mathcal{T}(n)$), with maps from $\mathcal{S}(n)$ and F^*B. Hence there is a canonical comparison $(F^*B)[n] \to F^*(B[n])$ from the free extension on F^*B.

Now take $\mathcal{S}$ an algebraic theory and A a model of $\mathcal{S}$. There is an algebraic theory $\mathcal{S}_A$ whose algebras are essentially $\mathcal{S}$-algebras equipped with a $\mathcal{S}$-algebra map from A. The precise formulation specifies $\mathcal{S} \to \mathcal{S}_A$ inducing a map $\mathrm{Alg}(\mathcal{S}_A) \to \mathrm{Alg}(\mathcal{S})$ and an isomorphism between A and $\mathcal{S}_A(0)$ regarded as an $\mathcal{S}$-algebra. Note in passing that there is a very simple description of $\mathcal{S}_A$ in case $A = \mathcal{S}(p)$ the free on p: we have $\mathcal{S}_{\mathcal{S}(p)}(n) = \mathcal{S}(n+p)$ with action not affecting the parameters in p.

It will be important to know when a map $\mathcal{S} \to \mathcal{T}$ of algebraic theories is of the form $\mathcal{S} \to \mathcal{S}_A$ for A an S-algebra. We have than an equivalence between $\mathrm{Alg}(\mathcal{S}_A) \to \mathrm{Alg}(\mathcal{S})$ and $A/\mathrm{Alg}(\mathcal{S}) \to \mathrm{Alg}(\mathcal{S})$ the forgetful from the co-clice category. The latter preserves connected colimits so pushouts. Hence for $A \to B$ in $A/\mathrm{Alg}(\mathcal{S})$ and we get pushout diagrams

$$\begin{tikzcd}
\mathcal{S}(0) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & B \arrow[d] \\
\mathcal{S}(n) \arrow[r] & A[n] \arrow[r] & B[n]
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{S})$, the right hand square coming from a pushout in $\mathrm{Alg}(\mathcal{S}_A)$. So the comparison $(F^*B)[n] \to F^*(B[n])$ is an isomorphism. Conversely suppose that $F : \mathcal{S} \to \mathcal{T}$ has comparison maps isomorphisms for all free extensions $\mathcal{T}(n)$ of $\mathcal{T}(0)$. Then $\mathcal{T}(n)$ is the free extension by n of $A = \mathcal{T}(0)$ and it is immediate that $\mathcal{T}$ is isomorphic to $\mathcal{S}_A$.

Proposition 2.5 *A map of algebraic theories $F : \mathcal{S} \to \mathcal{T}$ is isomorphic to one of the form $\mathcal{S} \to \mathcal{S}_A$ for a $\mathcal{S}$-algebra A if and only if all comparison maps $(F^*B)[n] \to F^*(B[n])$ are isomorphisms.*

A concrete way to think about this situation is that $\mathcal{S}_A$ is obtained from $\mathcal{S}$ by adding constants for the new elements of A and equations between constants but no other equations.

2.3 The presheaf topos

A $\mathcal{T}$-algebra is a set A equipped with an action of $\mathcal{T}$ on the left. But $\mathcal{T}$ can also act on the right. In the operad literature one talks confusingly of a module: I prefer to say presheaf.

Definition 2.6 *Let $\mathcal{T}$ be an algebraic theory. A* presheaf *X over $\mathcal{T}$ is a functor $X : \mathbf{F} \to \mathbf{Sets}$ equipped with an action $X(m) \times \mathcal{T}(n)^m \to X(n)$ compatible with the operations of $\mathcal{T}$. A map of presheaves is a functor commuting with the action of $\mathcal{T}$. The category of presheaves over $\mathcal{T}$ is denoted $P(\mathcal{T})$.*

Evidently $\mathcal{T}$ is itself a presheaf. It is natural to call it the generic presheaf though except in rare cases $P(\mathcal{T})$ is not the classifying topos for $\mathcal{T}$ in the topos theoretic sense. It is very easy to see that $P(\mathcal{T})$ is a category with products (indeed limits) defined pointwise. In particular there are objects $\mathcal{T}^m$, the finite powers of the generic, with $\mathcal{T}^m(n) = \mathcal{T}(n)^m$. A Yoneda Lemma argument gives the following.

Proposition 2.7 *We have $P(\mathcal{T})(\mathcal{T}^m, X) \cong X(m)$ natural in $X \in P(\mathcal{T})$.*

This shows that the products of the generic are dense, and so familiar arguments allow one to deduce more structure.

Proposition 2.8 *$P(\mathcal{T})$ is topos; in particular it is a locally cartesian closed category.*

For the λ-calculus, we are interested in function spaces. The following is essentially an old observation of Lawvere's.

Proposition 2.9 *For any presheaf X, the function space $\mathcal{T}^p \Rightarrow X$ is given by $(\mathcal{T}^p \Rightarrow X)(m) = X(m+p)$ with action $X(m+p) \times \mathcal{T}(n)^m \to X(n+p)$ leaving the parameters p undisturbed.*

The function spaces $\mathcal{T}^p \Rightarrow \mathcal{T}$ with $(\mathcal{T}^p \Rightarrow \mathcal{T})(m) = \mathcal{T}(m+p)$ will be of particular interest for the λ-calculus.

2.4 Monads and Lawvere Theories

In this paper I use a multicategory theory approach to algebraic theories. I regard it as the fundamental one and it is particularly suited to the λ-calculus. However for those who may prefer them I comment briefly on two more traditional categorical approaches to algebra.

An algebraic theory $\mathcal{T}$ induces a monad T on **Sets**. whose functor part is given by the coend formula $T(A) = \int^n \mathcal{T}(n) \times A^n$. Concretely $T(A)$ is the set of terms from $\mathcal{T}$ with constants from A replacing variables. The unit and multiplication of the monad correspond to constants and substitution. Conversely from a monad T we get an algebraic theory $\mathcal{T}$ where $\mathcal{T}(n) = T(n)$, since terms in n variables form the underlying set of the free algebra on a set of size n. The monads which arise from algebraic theories are exactly the finitary monads and there is an equivalence of categories between algebraic theories and finitary monads. The notions of algebra correspond. On the other hand the presheaf topos which is fundamental to my approach to the λ-calculus is less easy to handle from the monad point of view.

Lawvere theories introduced in [18] provide the other established approach to algebra. The equivalence with finitary monads was evident from the start but Lawvere theories have been comparatively neglected. That seems a mistake: [14] compares the two approaches with applications to computer science in mind. The Lawvere Theory corresponding to an algebraic theory is essentially the free category $\mathbf{T}$ with products generated by $\mathcal{T}$. Algebras are given by product preserving functors $\mathbf{T} \to \mathbf{Sets}$ and almost by definition this notion coincides with the one above. For the presheaf category there is the following easy consequence of Proposition 2.7.

Proposition 2.10 *The presheaf categories $P(\mathcal{T})$ and $P(\mathbf{T}) = [\mathbf{T}^{\mathrm{op}}, \mathbf{Sets}]$ are equivalent.*

Thus one could perfectly easily rewrite this paper from the Lawvere theory perspective.

3 The lambda calculus

3.1 Interpretation as theory

The underlying philosophy is to take seriously the basic syntax of term formation in context first made prominent in Martin-Löf's treatment of Type

Theory. The term formation rules are of form

$$\frac{\Gamma \vdash t \in a \Rightarrow b \quad \Gamma \vdash u \in a}{\Gamma \vdash tu \in b} \qquad\qquad \frac{\Gamma,\ x \in a \vdash s \in b}{\Gamma \vdash \lambda x.s \in a \Rightarrow b}$$

while the computation rule is $(\lambda x.s)u = s[u/x]$. The rules force one to handle terms in context and it is best to do that directly. We can capture the above by saying that an interpretation of the lambda calculus is a cartesian multicategory with semi-closed structure. The pure lambda calculus corresponds to the one object version.

Definition 3.1 *To equip an algebraic theory $\mathcal{L}$ with* semi-closed structure *is to give retractions $\mathcal{L}(n) \to \mathcal{L}(n+1) \to \mathcal{L}(n)$ natural in n and compatible with the actions $\mathcal{L}(m) \times \mathcal{L}(n)^m \to \mathcal{L}(n)$ and $\mathcal{L}(m+1) \times \mathcal{L}(n)^m \to \mathcal{L}(n+1)$. A* λ-theory *is an algebraic theory $\mathcal{L}$ equipped with semi-closed structure.*

To start with one should understand the definition quite concretely. The image of $\mathrm{id} \in \mathcal{L}(1)$ under the retraction $\mathcal{L}(1) \to \mathcal{L}(2)$ is a binary operation $\mathrm{app} \in \mathcal{L}(2)$ of application, traditionally denoted by concatenation $\mathrm{app}(x, y) = xy$. By convention application associates so that $abcd = ((ab)c)d$. Naturality implies that the retraction $\mathcal{L}(n) \to \mathcal{L}(n+1)$ is given by $a \to az$ where I adopt now and henceforth the convention to use z to denote the new extra variable in this and similar cases. The naturality of the section $\mathcal{L}(n+1) \to \mathcal{L}(n)$ is more subtle. The critical fact is that $\mathcal{L}(2) \to \mathcal{L}(1)$ does not generally take application app to the identity id. Rather it goes to an element $\mathbf{1}z$ where $\mathbf{1} \in \mathcal{L}(0)$ is the image of app in $\mathcal{L}(0)$. I extend that observation in a couple of ways. First, if $s \in \mathcal{L}(n+1)$ has image $\bar{s} \in \mathcal{L}(n)$, then we have $s = \bar{s}z$ and $\bar{s} = \mathbf{1}\bar{s}$. Secondly suppose that $s \in \mathcal{L}(n)$ has image $\hat{s} \in \mathcal{L}(0)$. Then with obvious notation $s = \hat{s}z_1 \cdots z_n$, and $\mathbf{1}_n\hat{s} = \hat{s}$ where $\mathbf{1}_n$ is the image in $\mathcal{L}(0)$ of iterated application $xz_1 \cdots z_n$ with n arguments.

It is clear how to interpret terms of the λ-calculus in a λ-theory: we have seen application and the sections $\mathcal{L}(n+1) \to \mathcal{L}(n)$ model λ-abstraction. For example $\lambda x, y.xy = \mathbf{1}$. Little more is needed to confirm that indeed β-equality is satisfied. The main point concerns substitution. One shows by induction on the structure of a term r that the interpretation of the substitution $r[s/x]$ of s for a variable x in r is implemented by composition of the interpetations for r and s. With that in place checking β-equality is straightforward. One can make this more precise using the natural notion of map of λ-theories.

Definition 3.2 *Let $\mathcal{L}$ and $\mathcal{M}$ be λ-theories. A* map $\mathcal{L} \to \mathcal{M}$ of λ-theories *is a map of algebraic theories which commutes with retraction and section. That gives a category of λ-theories.*

Now the λ calculus presents the initial λ-theory Λ. Let $\Lambda(n)$ be the terms of the λ-calculus in context of n variables, factored out by β-equality. Identities and projections are evident and composition is given by substitution. The retraction of $\Lambda(n)$ onto $\Lambda(n+1)$ is $t \in \Lambda(n) \mapsto t.z \in \Lambda(n+1)$. The section is $s \in \Lambda(n+1) \mapsto \lambda z.s \in \Lambda(n)$. Simple computations confirm that this is a λ-theory and it is easy to show the following.

Proposition 3.3 *With the structure indicated, Λ is the initial λ-theory.*

A map of λ-theories is more than a map of algebraic theories: the preservation of the semi-closed structure is essential. However the following fact is suggestive.

Proposition 3.4 *Suppose that $\mathcal{L}$ and $\mathcal{M}$ are λ-theories and we are given maps $\Lambda \to \mathcal{L} \to \mathcal{M}$ of algebraic theories. If the maps from Λ are the unique λ-theory maps then the map $\mathcal{L} \to \mathcal{M}$ is a map of λ-theories.*

Proof: Under the given conditions $\mathcal{L} \to \mathcal{M}$ carries application to application and preserves the combinator **1**. And that fixes all the structure.

3.2 Extensionality

I turn aside for a moment to comment on two separate issues termed extensionality. The more problematic[3] is usually called weak extensionality. As Scott [20] makes clear from the category theoretic perspective this is just the question of whether one has enough points. That makes sense also from the multicategory perspective. An algebraic theory $\mathcal{T}$ *has enough points* just when equality on each $\mathcal{T}(n)$ is reflected in the action $\mathcal{T}(n) \times \mathcal{T}(0)^n \to \mathcal{T}(0)$ on constants. Of course λ-theories may not have enough points: indeed the initial λ-theory does not. One should not worry about that. For those who do, there is this general fact.

Proposition 3.5 *Any algebraic theory $\mathcal{T}$ or λ-theory $\mathcal{L}$ embeds in an algebraic theory or λ-theory with enough points.*

[3] Note Barendregt's faintly apologetic remark 'In spite of not being weakly extensional λ-algebras are worth studying' on page 87 of [2].

Proof: For $\mathcal{T}$, take $\mathcal{T}(\omega)$ to be the free algebra on countable many generators, constructed in the obvious way as a direct limit of the free algebras $\mathcal{T}(n)$. Then $\mathcal{T}$ embeds in $\mathcal{T}_{\mathcal{T}(\omega)}$, the theory of extensions of $\mathcal{T}(\omega)$ which clearly has enough points. Similarly $\mathcal{L}$ embeds in $\mathcal{L}_{\mathcal{L}(\omega)}$ with enough points and it is easy to see that this gives an extension of λ-theories.

Of course this observation is not new for the λ-calculus: it is essentially in Section 4 of [3]. A special case is the move from the initial closed term to the open term interpretation.

Another quite different aspect of extensionality is the η-rule $\lambda x.tx = t$ when x is not free in t. This corresponds exactly to the requirement that the semi-closed structure be closed, in the sense that the retractions $\mathcal{T}(n) \to \mathcal{T}(n+1)$ are isomorphisms. In terms of the analyis above this is equivalent to application app being taken to the identity. In terms of combinators that is just $\mathbf{I} = \mathbf{1}$. The theory outlined in this paper is not much affected by adding this condition but doing so confuses rather than helps. There can be no doubt that it is the basic calculus with just the β-rule which is fundamental.

3.3 Reflexive objects

Scott's elegant categorical approach to the semantics of the λ-calculus stands out as clean mathematics. It will not do as a definition, but it captures much of importance in what follows. The most obvious element is a Representation Theorem. I give a new proof: the λ-theory perspective makes Scott's idea is easy to understand. Suppose that $\mathbf{C}$ is a cartesian closed category and U an object of $\mathbf{C}$ equipped with a retraction onto the function space $U^U = U \Rightarrow U$. Set $\mathcal{U}(n) = \mathbf{C}(U^n, U)$. This is automatically an algebraic theory: one might call it the endomorphism theory of U. Moreover since $\mathbf{C}(U^n, U^U) \cong \mathbf{C}(U^{n+1}, U)$ we get retractions from $\mathcal{U}(n)$ onto $\mathcal{U}(n+1)$ manifestly natural in n. So we get a λ-theory $\mathcal{U}$.

The essence of Scott's wonderful insight was that any λ-theory can be so represented. The proof with discussion of the significance of the result is in [20]. Serious coding is involved: Scott is proving more than the Represntation Theorem, and we come to that more in Section 4.2.

Theorem 3.6 (Scott's Representation Theorem) *Any λ-theory is isomorphic to the λ-theory of a reflexive object in a cartesian closed category.*

Proof: We prepared for this proof in section 2.3. Given $\mathcal{L}$ take the presheaf topos $P(\mathcal{L})$ and let U be the generic object $\mathcal{L}$ itself. In Proposition 2.9, we

computed the function space U^U: it is the presheaf $\mathcal{L}(n+1)$. By definition a λ-theory consists of a retract $\mathcal{L}(n)$ onto $\mathcal{L}(n+1)$ and the naturality conditions say that this is in the category $P(\mathcal{L})$. So we have a retract from U to U^U and U is a reflexive object. It remains to consider the λ-theory $\mathcal{U}$ obtained from U. For that we have $\mathcal{U}(n) = P(\mathcal{L})(U^n, U) \cong \mathcal{L}(n)$ by Proposition 2.7. It is evident that this is an isomorphism of λ-algebras.

The proof of the Representation Theorem given by Scott [20] for the Curry Festschrift is elementary and explicit and naturally gives more information: famously the reflexive object can be found in the cartesian closed category of retracts of a λ-algebra. In terms of λ-theories, what Scott does is the following. He considers the monoid $\mathcal{L}(1)$ as a one object category. Now any category $\mathbf{C}$ has a category of retract $\mathcal{K}\mathbf{C}$, called variously the Karoubi envelope or Cauchy completion of $\mathbf{C}$. Its objects are idempotents $e : A \to A$ with $e \circ e = e$ in $\mathbf{C}$; and maps between idempotents $e : A \to A$ and $f : B \to B$ are maps $v : A \to B$ such that $f \circ v \circ e = v$. Composition is inherited from $\mathbf{C}$ and each idempotent e is its own identity. Scott simply considered the Cauchy completion $\mathcal{K}\mathcal{L}(1)$ and showed by explicit calculation that it is cartesian closed. I shall use the presheaf category $P(\mathcal{L})$ to give a new proof.

3.4 The Taylor Fibration

In his PhD thesis [23], Paul Taylor extended Scott's analysis in a very remarkable way. Taylor observed that the category $\mathbf{R} = \mathcal{K}\mathcal{L}(1)$ is not just cartesian closed but is relatively cartesian closed in an interesting way. For each object $E \in \mathbf{R}$, Taylor localised the construction of $\mathbf{R}$ as follows. Let $U \in \mathbf{R}$ be the generating object given by the identity idempotent. Over $E \in \mathbf{R}$ we have $\Delta_E(U) = (U \times E \to E)$ which is a reflexive object in the slice $\mathbf{R}/E$. Taylor considered the subfibration $\mathbf{R}(E)$ of $\mathbf{R}/E$ consisting of retracts $A \to E$ of $\Delta_E(U)$. His result is that for every $\alpha : F \to E$ in $\mathbf{R}(E)$ the pullback functor $\alpha^* : \mathbf{R}(E) \to \mathbf{R}(F)$, which is evident from the definition of $\mathbf{R}(E)$, comes equipped with a right adjoint $\Pi_\alpha : \mathbf{R}(F) \to \mathbf{R}(E)$.

In the spirit of Scott [20], Taylor in [23] simply wrote down the various combinators and calculated to show that they work. I shall give a more abstract proof and also show that the pullback $\alpha^* : \mathbf{R}(E) \to \mathbf{R}(F)$ above has a left adjoint $\Sigma_\alpha : \mathbf{R}(F) \to \mathbf{R}(E)$. Scott's result that $\mathbf{R}$ is cartesian closed is a simple consequence.

To give my proof I use the presheaf category $P(\mathcal{L})$ with its generic object $U = \mathcal{L}$. I need just one piece of elementary λ-calculus. We already saw the retract from U to U^U. We need also a retract from U onto $U \times U$. It is very familiar: we have the inclusion

$$\mathcal{L}(n) \times \mathcal{L}(n) \to \mathcal{L}(n); \quad (a, b) \to \lambda x.xab \,,$$

and the retraction

$$\mathcal{L}(n) \to \mathcal{L}(n) \times \mathcal{L}(n); \qquad c \to (c\mathbf{T}, c\mathbf{F}) \,.$$

(Here $\mathbf{T} = \lambda x, y.x$ and $\mathbf{F} = \lambda x, y.y$.)

Now let us consider Taylor's fibration over the whole of $P(\mathcal{L})$. Thus for $X \in P(\mathcal{L})$, take $\mathbf{R}(X)$ to be the category of retracts of the reflexive object $\Delta_X(U) = (U \times X \to X)$ in $P(\mathcal{L})/X$.

Theorem 3.7 *Take $X \in P(\mathcal{L})$. Let $\alpha : Y \to X$ be in $\mathbf{R}(X)$. Then for any $Q \to Y$ in $\mathbf{R}(Y)$,*
(i) the standard indexed product $\Pi_\alpha Q \to X$ from $X \in P(\mathcal{L})$, lies in $\mathbf{R}(X)$; and
(ii) the standard indexed sum $\Sigma_\alpha Q \to X$ from $X \in P(\mathcal{L})$, lies in $\mathbf{R}(X)$.

Proof: For (i) we argue using a long chain of simple retractions. First as $Q \to Y$ is a retract of $\Delta_Y(U)$ it follows that $\Pi_\alpha Q \to X$ is a retract of $\Pi_\alpha \Delta_Y(U)$. So it is enough to consider the latter. Secondly since $\alpha : Y \to X$ is a retract of $\Delta_X(U)$ some elementary categorical computation shows that $\Pi_\alpha \Delta_Y(U)$ is a retract of $\Pi_q \Delta_{U \times X}(U)$ where $q : U \times X \to X$ is the second projection. But very easily $\Pi_q \Delta_{u \times X}(U) \cong \Delta_X(u \Rightarrow u)$. Trivially as u is reflexive so is $\Delta_X(U)$, so that $\Delta_X(U \Rightarrow U)$ is a retract of $\Delta_X(U)$. Combining the series of retracts we find that $\Pi_\alpha Q$ is a retract of $\Delta_X(U)$ and so is in $\mathbf{R}(X)$ as required.
Turning now to (ii) $\Sigma_\alpha Q \to X$ is just the composite of $Q \to Y$ with $\alpha : Y \to X$. Again $Q \to Y$ is a retract of $\Delta_Y(U)$ so we can concentrate on the latter. But as $\alpha : Y \to X$ is a retract of $\Delta_X(U)$ we see that $\Sigma_\alpha \Delta_Y(U)$ is a retract of $\Sigma_q \Delta_{U \times X}(U)$ where again $q : U \times X \to X$ is the second projection. But that is isomorphic to $\Delta_X(U \times U)$; and as $U \times U$ is a retract of U we again compose retractions to find $\Sigma_\alpha Q$ a retract of $\Delta_X(U)$ and so is in $\mathbf{R}(X)$.

Corollary 3.8 (Taylor) *The category of retracts of a λ-theory is relatively cartesian closed.*

Proof: The category $\mathbf{R}$ of retracts of U is a subcategory of $P(\mathcal{L})$. Furthermore if $X \in \mathbf{R}$ then so are the objects of $\mathbf{R}(X)$. So the result is immediate by restricting to $\mathbf{R}$.

Corollary 3.9 *The category of retracts of a λ-theory is cartesian closed.*

Proof: Immediate by restricting to the fibre $\mathbf{R}(1)$ over 1.

I remarked earlier that for a general algebraic theory $\mathcal{T}$, the presheaf category $P\mathcal{T}$ is equivalent to presheaves on the Lawvere theory generated by $\mathcal{T}$. For λ-theories $\mathcal{L}$ we have much more as the monoid $\mathcal{L}(1)$ and the category of retracts $\mathbf{R}$ are Morita equivalent, and the Lawvere theory $\mathbf{L}$ is caught between them.

Proposition 3.10 *The categories of presheaves $P(\mathcal{L}(1))$, $P(\mathcal{L})$ $P(\mathbf{L})$ and $P(\mathbf{R})$ are all equivalent.*

4 Algebras

4.1 Algebras for λ-theories

Consider any λ-theory $\mathcal{L}$ simply as an algebraic theory, and then we have a category $\mathrm{Alg}(\mathcal{L})$ of $\mathcal{L}$-algebras. A $\mathcal{L}$-algebra is a set A equipped with actions $\mathcal{L}(n) \times A^n \to A$. Concretely that means that for each term $t(\mathbf{x})$ in $\mathcal{L}(n)$ and each n-tuple $\mathbf{a} \in A^n$ we get an interpretation $t(\mathbf{a})$ in A; and this behaves as expected on variables and respects substitution and β-equality.

It turns out that one can focus almost entirely on Λ, the initial λ-theory. A Λ-algebra is a set A equipped with good interpretations $t(\mathbf{a})$ in A for every term $t(\mathbf{x})$ of the λ-calculus and n-tuple $\mathbf{a} \in A^n$. That is a clean definition close in spirit to environment or valuation models but avoiding the explicit interpretation of abstraction and so the vexed question of weak extensionality. Giving Λ-algebra structure amounts to giving λ-algebra structure as in [1], but to underline the difference in perspective, I use the category theoretic terminology and refer to Λ-algebras.

Any λ-theory $\mathcal{L}$ gives rise to a Λ-algebra in a straightforward way. Indeed suppose we have any map $\Lambda \to \mathcal{T}$ of algebraic theories. This induces a functor from $\mathrm{Alg}(\mathcal{T})$ to $\mathrm{Alg}(\Lambda)$. In particular there is an induced structure of Λ-algebra on the initial $\mathcal{T}$-algebra $\mathcal{T}(0)$. This is evidently functorial in $\mathcal{T}$.

If we consider a λ-theory $\mathcal{L}$, we can specialise to the a unique map $\Lambda \to \mathcal{L}$ of λ-theories and we can apply the above to get a Λ-algebra $\mathcal{L}(0)$ in that case.

Proposition 4.1 *Passing from $\Lambda \to \mathcal{T}$ to $\mathcal{T}(0)$ gives a functor from the co-slice of algebraic theories under Λ to* $\mathrm{Alg}(\Lambda)$. *This restricts to a functor from λ-theories to* $\mathrm{Alg}(\Lambda)$.

4.2 Presheaves on the monoid

Now I recall the essence of Scott's analysis in [20] to give a construction producing a λ-theory from a Λ-algebra.

Let A be a Λ-algebra. On $A(1) = \{a \in A | \mathbf{1}a = a\}$ take the monoid structure with multiplication[4] $(a, b) \mapsto a \circ b = \lambda x.a(bx)$ representing composition. The underlying set is a retract of A and another way to think of it is as formal elements of the form az factored out by $a \sim b$ if and only if $\mathbf{1}a = \mathbf{1}b$ and with composition $(az, bz) \mapsto a(bz) = (a \circ b)z$. Write M_A for this monoid. Now consider $P(A) = P(M_A)$ the category of presheaves on the monoid or one object category M_A. The generic object is $U = M_A$ as underlying set with the evident right action of $b \in M_A$ by composition, $(a, b) \mapsto a \circ b$. My approach to the construction is to identify the function space U^U.

I start by giving the categorical analysis for a general monoid M. Let U be the generic object in $P(M)$. The standard calculation gives U^U equal to $P(M)(U \times U, U)$ with action in the first copy of U. That amounts to the following data in the basic theory of presheaf toposes. For each $p \in M$ we have a map $\alpha_p : M \to M$ with the naturality property that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\alpha_p} & M \\ {\scriptstyle -\circ m}\downarrow & & \downarrow{\scriptstyle -\circ m} \\ M & \xrightarrow[\alpha_{p\circ m}]{} & M \end{array}$$

commutes for all $m \in M$; that is, $\alpha_p(q) \circ m = \alpha_{p \circ m}(q \circ m)$. Generally that is all there is to say, but for $M = M_A$ there is more.

[4]The underlying set is a retract of A and another way to think of it is as formal elements of the form az factored out by $a \sim b$ if and only if $\mathbf{1}a = \mathbf{1}b$ and with composition $(az, bz) \mapsto a(bz) = (a \circ b)z$. That makes clear the connection with the monoid $\mathcal{L}(1)$ used earlier.

Take $p = \lambda x.x\mathbf{T}$ and $q = \lambda x.x\mathbf{F}$ with $\mathbf{T}$ and $\mathbf{F}$ as before. $\alpha_p(q)$ is in a sense generic. For any a, b we can consider $m = \lambda x.xab$ and we see that $\alpha_p(q) \circ m = \alpha_a(b)$. So if we take

$$d = \lambda y, z.(\alpha_p(q) \circ (\lambda x.xyz)) = \lambda y, z, w.\alpha_p(q)(wyz)$$

we have an element $d \in A(2) = \{d | \lambda x, y.dxy = d\} = \{d | \mathbf{1} \circ d = d\}$ representing α in the sense that $\alpha_a(b) = \lambda y.d(ay)(by)$. It is easy to check that any such d uniquely represents an α. Thus we can take U^U to be the set $A(2)$ with evident action by composition. Obviously composing on the left with $\mathbf{1}$ gives a retract from U to U^U and the generic U is a reflexive object in the presheaf category $P(A)$.

The reflexive $U \in P(A)$ gives us a λ-theory $\mathcal{U}_A$ which we could call the universal λ-theory of A. We shall need some easy functoriality. If $A \to B$ is a map of Λ-algebra, then we get an induced map $M_A \to M_B$ of monoids and so by left Kan extension an induced functor $P(M_A) \to P(M_B)$ which evidently takes the universal U_A in $P(A)$ to U_B in $P(B)$. The left Kan extension preserves products and so action of the functor gives maps

$$P(A)(U_A^n, U_A) \to P(B)(U_B^n, U_B)$$

which form a map of algebraic theories $\mathcal{U}_A \to \mathcal{U}_B$. All the idempotents from the monoid which we have just exploited are λ-definable and so preserved by $M_A \to M_B$; and splitting idempotents is universal so the iterated function spaces are taken to iterated function spaces and that is enough to show we have a map of λ-theories.

Proposition 4.2 *The operation taking A to $\mathcal{U}_A$ gives a functor from Λ-algebras to λ-theories.*

For future reference I say just a bit more about the

$$\mathcal{U}_A(n) = P(A)(U^n, U) \cong P(A)(1, U^n \Rightarrow U)\,.$$

An easy extension of the argument above shows that each $U^n \Rightarrow U$ can be represented by $A(n+1) = \{d | \mathbf{1}_n \circ d = d\}$ with again the obvious action. Under evaluation $d \in A(n+1)$ corresponds to

$$(a_1, \cdots, a_n) \mapsto \lambda y.d(a_1 y) \cdots (a_n y)$$

as a map $U^n \to U$. Observe that generally

$$\{d \in M_A | \mathbf{1}_n \circ d = d\} = \{d \in A | \mathbf{1}_{n+1} d = d\}\,.$$

where the left hand side has a clear action on the right by the monad M_A. Taking points or global sections is taking fixed points under the action and an easy argument with constants shows that gives $A(n)$. Thus

$$\mathcal{U}_A(n) \cong A(n) = \{a \in A | \mathbf{1}_n a = a\}\,.$$

4.3 The Fundamental Theorem

Let us now extract information from the construction. It is counter-intuitive but it is best to start with a λ-theory $\mathcal{L}$. The initial $\mathcal{L}$-algebra $\mathcal{L}(0)$ is a Λ-algebra and I claim that we have an isomorphism $\mathcal{U}_{\mathcal{L}(0)} \cong \mathcal{L}$ of λ-theories. This is immediate from Section 3.4. The monoids $M_{\mathcal{L}(0)}$ and $\mathcal{L}(1)$ are canonically isomorphic and we know that the presheaf topos $P(\mathcal{L}(1))$ is equivalent to $P\mathcal{L}$ in such a way that the generic objects U corresponds. But by the above $\mathcal{U}_{\mathcal{L}(0)}$ can be defined as the λ-theory generated by U in the first case; and $\mathcal{L}$ is the λ-theory generated by U in the second.

In the other direction, suppose we take a Λ-algebra A, pass to the universal λ-theory $\mathcal{U}_A$ and then take the induced Λ-algebra $\mathcal{U}_A(0)$. It is the crux of Scott's analysis in [20] that these are isomorphic, but he passes over this rather quickly[5] so I just explain what he means. Inductively we consider open terms in the λ-calculus with constants from A and we show that each term $t(\mathbf{x})$ with n free variables is interpreted in $\mathcal{U}_A(n) \cong A(n)$ as above by the interpretation on A of its closure $\lambda\mathbf{x}.t(\mathbf{x})$. As Scott indicates that is completely straightforward, but I offer an alternative in the next section.

Proposition 4.3 *The functors $\mathcal{L} \mapsto \mathcal{L}(0)$ and $A \mapsto \mathcal{U}_A$ give an equivalence of categories between the categories of λ-theories and A a Λ-algebras.*

That is almost but not quite what I want to call the Fundamental Theorem. That comes from identifying $\mathcal{U}_A$ with Λ_A, the theory of extensions of A. Note that it is not clear a priori that the algebraic theory Λ_A is a λ-theory nor in such a way that $\Lambda \to \Lambda_A$ is a map of such. But we can deduce this from the

[5]Scott says simply 'As U is in fact the identity function, the reinterpretation via U of the type-free calculus will obviously translate every term into itself

equivalence of categories. We have the category $\mathrm{Alg}(\mathcal{U}_A)$ and taking A as the initial in the category, a forgetful functor from to $A/\mathrm{Alg}(\Lambda)$ the co-slice under A taken in Λ-algebras. It is obviously faithful. To show it is full, regard $A \to B \to C$ as a map in the co-slice. That induces $\mathcal{U}_A \to \mathcal{U}_B \to \mathcal{U}_C$ in λ-theories and thus the map from $B \cong \mathcal{U}_B(0)$ to $C \cong \mathcal{U}_C(0)$ is even a map of $\mathcal{U}_B$-algabras and so certainly $\mathcal{U}_A$-algebras. The same argmuent shows that our forgetful is essentially surjective on objects. If $A \to B$ is a map of Λ-algebras consideration of $\mathcal{U}_A \to \mathcal{U}_B$ shows it is a map of $\mathrm{S}\mathcal{U}_A$-algebras.

Theorem 4.4 (Fundamental Theorem of the λ-Calculus) *For $\mathcal{L}$ a λ-theory and A a Λ-algebra there are natural isomorphisms $\mathcal{L} \cong \Lambda_{\mathcal{L}(0)}$ and $A \cong \Lambda_A(0)$ so that we have an equivalence of categories between that of λ-theories and that of Λ-algebras. In particular λ-theory $\mathcal{L}$ is isomorphic to the λ-theory $\Lambda_{\mathcal{L}(0)}$ whose algebras are extensions of $\mathcal{L}(0)$*

What this formulation makes clear is that we are always in the situation of Proposition 2.5. There are many straightforward consequences. Note in particular that for every λ-theory $\mathcal{L}$, the $\mathcal{L}(n)$ are not just the free extensions of $\mathcal{L}(0)$ as $\mathcal{L}$-algebras but also as Λ-algebras. Much the same thought is expressed in the following.

Proposition 4.5 *Suppose that A is a Λ-algebra. Then there is a canonical Λ-algebra stucture on the retracts $A(n) = \{a \in A | \mathbf{1}_n a = a\}$ making them the free Λ-algebra extending A by n indeterminates.*

Proof: Immediate given the discussion following Proposition 4.2.

This last result is folklore mentioned in passing in [12] and spelled out in [21]. In the approach I take it emerges as a central property of the λ-calculus.

4.4 Alternative approaches

I can readily imagine experts who will not be wholely comfortable with aspects of my approach to the Fundamental Theorem, so I briefly sketch some alternatives.

First I offer a curious way to avoid Scott's inductive argument in the last section. Suppose that A is an algebra for a λ-theory $\mathcal{L}$. We have actions $\mathcal{L}(m) \times A^m \to A$ satisfying the standard conditions. Now $\mathcal{L}(0)$ is the initial $\mathcal{L}$-algebra so we have a unique map $\mathcal{L}(0) \to A$ of $\mathcal{L}$-algebras. Take $s \in \mathcal{L}(m)$ and $\mathbf{a} \in A^m$. It follows from the discussion following Definition 3.1 that

we can express the interpretation $s(\mathbf{a}) \in A$ in terms of the image $\hat{s}$ of s in $\mathcal{L}(0)$ using iterated application: $s(\mathbf{a}) = \hat{s}\mathbf{a}$. So the actions are completely determined by the simple map $\mathcal{L}(0) \to A$ and the application in A. Now write App for the category of applicative structures. We have the obvious forgetful functor $\mathrm{Alg}(\mathcal{L}) \to \mathrm{App}$. Suppose that A and B are $\mathcal{L}$-algebras and that we have maps $\mathcal{L}(0) \to A \to B$ with $\mathcal{L}(0) \to A$ and $\mathcal{L}(0) \to B$ the unique $\mathcal{L}$ algebra maps. It follows from the above that if $A \to B$ is a map of applicative structures then it is an $\mathcal{L}$-algebra map. We can encapsulate that in terms of the clearly faithful functor $\mathrm{Alg}(\mathcal{L}) \to \mathcal{L}(0)/\mathrm{App}$.

Proposition 4.6 *The functor* $\mathrm{Alg}(\mathcal{L}) \to \mathcal{L}(0)/\mathrm{App}$ *is full.*

But in the last section it is easy to see that the sets A and $\mathcal{U}_A(0)$ are isomorphic in such a way that the applications correspond. Further A is equipped with its unique map $\Lambda(0) \to A$; and we also get $\mathcal{U}_{\Lambda(0)} \cong \Lambda$ and so a unique map $\Lambda(0) \to \mathcal{U}_A(0)$. The maps from $\Lambda(0)$ factor through the isomorphism $Acong\mathcal{U}_A(0)$. So by Proposition 4.6 that is an isomorphism of Λ-algebras.

More radically one could construct Λ_A syntactically. Given a λ-algebra A, take an extension of the syntax of the λ-calculus with constants from A. Let $\Lambda_A(n)$ be the terms with n variables factored out by the equality generated by β-equality in the λ-calculus and by the equalities given by the actions $\Lambda(m) \times A^m \to A$. There is much to check: that we have an algebraic theory; that it is the theory of extensions of A; and that it is in fact a λ-theory, for which the argument is as for the initial λ-theory Λ. With that in place the functoriality of the construction is not hard.

Proposition 4.7 *Passing from a* Λ*-algebra* A *to the* λ*-theory* Λ_A *is functor from* Λ*-algebras to* λ*-theories.*

With this approach it is easy to see that given A we pass to Λ_A and an isomorphism $A \cong \Lambda_A(0)$ of Λ-algebras. Consider on the other hand $\Lambda \to \mathcal{T}$ a map of algebraic theories. An algebra for $\Lambda_{\mathcal{T}(0)}$ is a Λ-algebra with Λ-algebra map from $\mathcal{T}(0)$. So because an algebra for $\mathcal{T}$ canonically gives such data we have a map of algebraic theories $\Lambda_{\mathcal{T}(0)} \to \mathcal{T}$. It is then quite a lot of relatively straightforward work to check that this is the counit of an adjunction with unit the isomorphism $A \to \Lambda_A(0)$ above.

Proposition 4.8 *The functor taking* A *to* Λ_A *from* Λ*-algebras to theories under* Λ *has as left adjoint the functor taking* $\Lambda \to \mathcal{T}$ *to* $\mathcal{T}(0)$ *making* Λ*-algebras reflective in algebraic theories extending* Λ*.*

Finally we want to show that when we consider the unique map of λ-theories $\Lambda \to \mathcal{L}$, the counit above is an isomorphism. That amounts to showing that the $\mathcal{L}(n)$ are the free extensions of $\mathcal{L}(0)$ as Λ-algebras not simply as $\mathcal{L}$-algebras. For that there is an argument using Proposition 4.6 in a similar fashion to that above.

5 Conclusions and Vistas

Though this paper is foundational, I hope that it will encourage new research. I want here to stress that understanding λ-theories is understanding Λ-algebras. There is much more to that than the induced equality on pure λ-terms. Customary questions about the theories represented by various cartesian closed categories seem narrow: one should ask about the λ-theories represented. What can one say about them? How for example to compare them within and between categories? Note that Taylor's extension Theorem 3.4 of the Scott analysis gives a huge family of interpretations deriving from a specific one. But they by no means exhaust the interpretations to be derived from any presheaf category $P(\mathcal{L})$. Potentially that raises all sorts of questions.

A particularly interesting set of questions concerns cartesian closed categories arising from models of the differential lambda calculus. A form of the Approximation Theorem holds automatically and it follows that the quotients of the syntactic theories obtained are very restricted. These issues are intimately tied up with Böhm trees. For background see [8], [9] and [7]. However it appears that notwithstanding the restrictions, there is still a wide variety of interpretations. Is that true? How can one make sense of that? What general tools are there for telling differences in such cases?

I close with some remarks about potential wider significance of the techniques discovered by Corrado Böhm and presented in his seminal paper [6]. This is surely a cornerstone of our understanding of the λ-calculus and would now come early in any account of fundamental ideas. But how well do we understand what is involved? Prima facie the techniques are syntactic. The original applications are to quotients of the initial $\beta\eta$ theory Λ_η and concern what are usually thought of as the limits of consistency: what can you or can you not do before such a quotient becomes the terminal (trivial) theory? There is Böhm's original point that identifying distinct $\beta\eta$ normal forms collapses a theory and the closely related result of [13] that there is a unique

maximal non-trivial quotient of the theory extending Λ_η by setting all unsolvables equal. But I think there is a broader set of semantic principles at stake. For example when analysing models such as Scott's $P\omega$ in which η does not hold one exploits non-syntactic variants of Böhm's combinators. (This is well explained in section 3 of [4] in connection with Plotkin's T^ω model.) There must surely be more to be said on a broader canvas.

In honouring Böhm I have wanted to stress the future. I am sure that there is still much to discover about the pure λ-calculus; but I want the main message of this paper to be that it really is a remarkable and elegant area of mathematics.

References

[1] H. P. Barendregt. The Lambda Calculus: Its Syntax and Semantics.. Studies in Logic and the Foundations of Mathematics **103**, North-Holland 1981.

[2] H. P. Barendregt. The Lambda Calculus: Its Syntax and Semantics.. Studies in Logic and the Foundations of Mathematics **103**, Revised Edition, North-Holland 1984.

[3] Henk Barendregt and Karst Koymans. Comparing some classes of Lambda Calculus Models. In J. P. Seldin and J. R. Hindley editors. *To H. B. Curry: Essays on Combinatory Logic and Formalism*, Academic Press, 1980, 287-301.

[4] Henk Barendregt and Giuseppe Longo. Equality of Lambda Terms in the Model T^ω. In J. P. Seldin and J. R. Hindley editors. *To H. B. Curry: Essays on Combinatory Logic and Formalism*, Academic Press, 1980, 303-337.

[5] Nick Benton, Gavin Bierman, Valeria de Paiva and Martin Hyland. A Term Calculus for Intuitionistic Linear Logic. In M. Bezem and J.F. Groote (editors). Proceedings of the International Conference on Typed Lambda Calculi and Applications, TLCA 93. Lecture Notes in Computer Science 664, Springer-Verlag, 1993, 179-194.

[6] C. Böhm. Alcune proprietà delle forme β-η-normali nel $\lambda - K$-calcolo. Pubblicazioni dell' Istituto per le Applicazioni del Calcolo no **696**, Roma, 1968, (19pp).

[7] A. Carraro. Models and theories of pure and resource lambda calculi. PhD Thesis, Paris VII and Venezia, 2011.

[8] T. Ehrhard and L. Regnier. The differential lambda-calculus. Theoretical Computer Science **309**, 2003, 1-41.

[9] T. Ehrhard and L. Regnier. Böhm trees, Krivine machine and the Taylor expansion of ordinary lambda-terms. Lecture Notes in Computer Science **3988**, 2006, 186-197.

[10] M. Fiore, N.Gambino, M. Hyland and G. Winskel. Kleisli Bicategories. In preparation.

[11] M. Fiore, G. Plotkin and D. Turi. Abstract Syntax and Variable Binding. Proceedings 14th Annual IEEE Symposium on Logic in Computer Science, 1999, 193-202.

[12] P. J. Freyd. Combinators. In John W Gray and Andre Scedrov editors: Categories in Computer Science and Logic. Contemporary Mathematics **92**, AMS 1989, 63-66.

[13] M. Hyland. A syntactic characterisation of the Equality in some Models of the Lambda Calculus. Journal of the London Mathematical Society **12**, 1976, 361-370.

[14] Martin Hyland and John Power. The Category Theoretic Understanding of Universal Algebra: Lawvere Theories and Monads. In L. Cardelli, M. Fiore and G. Winskel (editors). *Computation, Meaning and Logic. Articles Dedicated to Gordon Plotkin.* Electronic Notes in Theoretical Computer Science 172, 2007, 437-458.

[15] C. P. J. Koymans. Models of the Lambda Calculus. PhD Thesis, Utrecht, 1984.

[16] J. Lambek. From λ-calculus to Cartesian Closed Categories. In J. P. Seldin and J. R. Hindley editors. *To H. B. Curry: Essays on Combinatory Logic and Formalism*, Academic Press, 1980, 375-402.

[17] J. Lambek and P.J. Scott. Introduction to higher order categorical logic. Cambridge University Press, 1986.

[18] F. William Lawvere. Functorial Semantics of Algebraic Theories. PhD Thesis, Columbia University, 1963. Republished with an introduction in Reprints in Theory and Applications of Categories, **5**, 2004, 1-121.

[19] G. Manzonetto and A. Salibra. Boolean algebras for λ-calculus. Proceedings 21st IEEE Symposium on Logic in Computer Science, 2006, 317-326.

[20] Dana S. Scott. Relating Theories of the λ-calculus. In J. P. Seldin and J. R. Hindley editors. *To H. B. Curry: Essays on Combinatory Logic and Formalism*, Academic Press, 1980, 403-450.

[21] Peter Selinger. The lambda calculus is algebraic. Journal of Functional Programming **12**, 549-566.

[22] M. Tanaka and J. Power: A Unified Category-theoretic Semantics for Binding Signatures in Substructural Logics. Journal Logic and Computation. **16**, 2006, 5-25.

[23] Paul Taylor. Recursive Domains, Indexed Category Theory and Polymorphism. PhD Thesis, University of Cambridge, 1986.